Quick Reference Notes

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Contents

1 Lambda calculi with types

1.1 Simply typed lambda calculus

- Simple types ($\mathcal{T}_{\rightarrow}$)
  - type variables $P_0, P_1, \ldots$.
  - if $A, B \in \mathcal{T}_{\rightarrow}$, then $A \rightarrow B \in \mathcal{T}_{\rightarrow}$
- Product type ($\mathcal{T}_{\rightarrow,\times}$)
  - if $A, B \in \mathcal{T}_{\rightarrow,\times}$, then $A \times B \in \mathcal{T}_{\rightarrow,\times}$
- Sums ($\mathcal{T}_{\rightarrow,\times,+}$)
  - if $A, B \in \mathcal{T}_{\rightarrow,\times,+}$, then $A + B \in \mathcal{T}_{\rightarrow,\times,+}$

Typing schemes:

$$\frac{}{\vdash x^A}$$

$$\frac{\vdash M^B}{\vdash (\lambda x^A.M^B)^{A\rightarrow B}} \qquad \frac{\vdash M^{A\rightarrow B} \quad \vdash N^A}{\vdash (M^{A\rightarrow B}\, N^A)^B}$$

$$\frac{\vdash M^A \quad \vdash M^B}{\vdash \langle M^A, N^B\rangle^{A\times B}} \qquad \frac{\vdash M^{A\times B}}{\vdash \pi_1(M^{A\times B})^A} \qquad \frac{\vdash M^{A\times B}}{\vdash \pi_2(M^{A\times B})^B}$$

$$\frac{\vdash M^A}{\vdash \texttt{inl}(M^A)^{A+B}} \qquad \frac{\vdash M^B}{\vdash \texttt{inr}(M^B)^{A+B}} \qquad \frac{\vdash M^{A+B} \quad \vdash P^C \quad \vdash Q^C}{\vdash (\delta\, M^{A+B}\, (x^A.P^C)\ (y^B.Q^C))^C}$$

Reduction: compatible closure of the following reduction axioms

$$
\begin{array}{lll}
(\lambda x^A.P^B)\,Q^A & \to_\beta & P^B\{x^A := Q^A\} \\
\lambda x^A.P^{A\supset B}\,x^A & \to_\eta & P, \quad (x \notin FV(P))
\end{array}
$$

$$
\begin{array}{lll}
\pi_1(\langle M, N\rangle) & \to_{\pi_1} & M \\
\pi_2(\langle M, N\rangle) & \to_{\pi_2} & N \\
\langle \pi_1(M), \pi_2(M)\rangle & \to_{SP} & M
\end{array}
$$

$$
\begin{array}{lll}
\delta\,\texttt{inl}(M)^{A+B}\,(x^A.N_1)\ (y^B.N_2) & \to_{+_1} & N_1\{x := M\} \\
\delta\,\texttt{inr}(M)^{A+B}\,(x^A.N_1)\ (y^B.N_2) & \to_{+_2} & N_2\{y := M\} \\
\delta\,M\,(x^A.\texttt{inl}(x)^{A+B})\ (y^B.\texttt{inr}(y)^{A+B}) & \to_{+_{SP}} & M
\end{array}
$$

1.2 Dependent types system λP

The set of *raw expressions*

$$
\begin{array}{lrcl}
\text{Raw environments} & \Gamma & ::= & \emptyset \,|\, \Gamma, x : A \,|\, \Gamma, \alpha : \kappa \\
\text{Raw kinds} & \kappa & ::= & * \,|\, \Pi x^A.\kappa \\
\text{Raw types} & A & ::= & \alpha \,|\, (\forall x^A.A) \,|\, (A\,M) \,|\, (\lambda x^A.A) \\
\text{Raw terms} & M & ::= & x^A \,|\, (M\,M) \,|\, (\lambda x^A.M)
\end{array}
$$

Notes:

- $A \Rightarrow \kappa$ abbreviation for $\Pi x : A.\kappa$, if $x \notin FV(\kappa)$.
- $A \to B$ abbreviation for $\forall x : A.B$, if $x \notin FV(B)$.

Reduction schemes: for C any raw expression with a hole of the appropriate sort (term/type):

$$
\begin{array}{c}
C[(\lambda x^A.P)\,Q] \to_\beta C[P\{x := Q\}] \\
C[(\lambda x^A.B)\,Q] \to_\beta C[B\{x := Q\}]
\end{array}
$$

Regarding the typing schemes of λP, there are three types of judgements

- $\Gamma \vdash \kappa : \Box$ (kind formation)
- $\Gamma \vdash A : \kappa$ (kinding)
- $\Gamma \vdash M : A$ (typing)

Kind formation schemes

$$\frac{}{\vdash * : \Box} \qquad \frac{\Gamma, x : A \vdash \kappa : \Box}{\Gamma \vdash \Pi x^A.\kappa : \Box}$$

Kinding schemes

$$\frac{\Gamma \vdash \kappa : \Box}{\Gamma, \alpha : \kappa \vdash \alpha : \kappa}\,(\alpha \notin FV(dom(\Gamma))) \qquad \frac{\Gamma, x : A \vdash B : *}{\Gamma \vdash \forall x^A.B : *}$$

$$\frac{\Gamma \vdash A : (\Pi x^B.\kappa) \quad \Gamma \vdash M : B}{\Gamma \vdash (A\,M) : \kappa\{x := M\}} \qquad \frac{\Gamma, x : A \vdash B : \kappa}{\Gamma \vdash (\lambda x^A.B) : (\Pi x^A.B)}$$

$$\frac{\Gamma \vdash A : \kappa \quad \Gamma \vdash \kappa' : \Box}{\Gamma \vdash A : \kappa'}\,(\kappa =_\beta \kappa')$$

Typing schemes

$$\frac{\Gamma \vdash A : *}{\Gamma, x : A \vdash x : A}\,(x \notin dom(\Gamma))$$

$$\frac{\Gamma \vdash M : (\forall x^A.B) \quad \Gamma \vdash N : A}{\Gamma \vdash (M\,N) : B\{x := N\}} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x^A.M) : (\forall x^A.B)}$$

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash A' : *}{\Gamma \vdash M : A'}\,(A =_\beta A')$$

Weakening schemes

$$\frac{\Gamma \vdash A : * \quad \Gamma \vdash \kappa : \Box}{\Gamma, x : A \vdash \kappa : \Box} \qquad \frac{\Gamma \vdash \kappa : \Box \quad \Gamma \vdash \kappa' : \Box}{\Gamma, \alpha : \kappa \vdash \kappa' : \Box}$$

$$\frac{\Gamma \vdash A : * \quad \Gamma \vdash A : \kappa}{\Gamma, x : A \vdash A : \kappa} \qquad \frac{\Gamma \vdash \kappa : \Box \quad \Gamma \vdash A : \kappa'}{\Gamma, \alpha : \kappa \vdash A : \kappa'}$$

$$\frac{\Gamma \vdash A : * \quad \Gamma \vdash M : A}{\Gamma, x : A \vdash M : A} \qquad \frac{\Gamma \vdash \kappa : \Box \quad \Gamma \vdash M : A}{\Gamma, \alpha : \kappa \vdash M : A}$$

2 Natural Deduction

2.1 Axiom and Inference Schemes

A ("can always prove A if I assume A!")

$$\frac{A \quad B}{A \wedge B}\wedge_I \qquad \frac{A \wedge B}{A}\wedge_{E_1} \quad \frac{A \wedge B}{B}\wedge_{E_2}$$

$$\frac{\begin{array}{c}[A]^u \\ \vdots \\ B\end{array}}{A \supset B}\supset_I, u \qquad \frac{A \supset B \quad A}{B}\supset_E$$

$$\frac{A}{A \vee B}\vee_{I_1} \quad \frac{B}{A \vee B}\vee_{I_2} \qquad \frac{A \vee B \quad \begin{array}{c}[A]^u \\ \vdots \\ C\end{array} \quad \begin{array}{c}[B]^v \\ \vdots \\ C\end{array}}{C}\vee_E, u, v$$

$$\frac{\bot}{A}\bot_I$$

$$\frac{A\{x := y\}}{\forall x.A}\forall_I \qquad \frac{\forall x.A}{A\{x := t\}}\forall_E$$

$$\frac{A\{x := t\}}{\exists x.A}\exists_I \qquad \frac{\exists x.A \quad \begin{array}{c}[A\{x := y\}]^u \\ \vdots \\ C\end{array}}{C}\exists_E, u$$

- Condition on $\forall_I$: $y \equiv x$ or $y \notin FV(A)$; and y not free in any open assumption of the derivation ending in $A\{x := y\}$
- Condition on $\exists_E$: $y \equiv x$ or $y \notin FV(A)$; and y not free in any open assumption of the derivation ending in C except for $A\{x := y\}$

Alternative Schemes for Obtaining **Nc** from **Ni**.

$$A \vee \neg A \qquad ((A \supset B) \supset A) \supset A$$

(Pierce)

$$\frac{\neg\neg A}{A}\neg_{E_1} \qquad \frac{\begin{array}{c}[A]^u \\ \vdots \\ B\end{array} \quad \begin{array}{c}[\neg A]^v \\ \vdots \\ B\end{array}}{B}\neg_{E_2}, u, v \qquad \frac{\begin{array}{c}[A \supset B]^u \\ \vdots \\ A\end{array}}{A}P, u$$

2.2 ND in Sequent Style

$$\frac{}{u : A \vdash A}\,Ax \qquad \frac{\Gamma, u : A, v : A \vdash B}{\Gamma, w : A \vdash B}\,Cont$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\,\wedge_I \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\,\wedge_{E_1} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\,\wedge_{E_2}$$

$$\frac{\Gamma, [u : A] \vdash B}{\Gamma \vdash A \supset B}\,\supset_I \qquad \frac{\Gamma \vdash A \supset B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B}\,\supset_E \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\,\vee_{I_1}$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\,\vee_{I_2} \qquad \frac{\Gamma \vdash A \vee B \quad \Delta, [u : A] \vdash C \quad \Theta, [v : B] \vdash C}{\Gamma, \Delta, \Theta \vdash C}\,\vee_E$$

$$\frac{\Gamma, [u : \neg A] \vdash \bot}{\Gamma \vdash A}\,\bot_C \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash A}\,\bot_I$$

$$\frac{\Gamma \vdash A\{x := y\}}{\Gamma \vdash \forall x.A}\,\forall_I \qquad \frac{\Gamma \vdash \forall x.A}{\Gamma \vdash A\{x := t\}}\,\forall_E$$

$$\frac{\Gamma \vdash A\{x := t\}}{\Gamma \vdash \exists x.A \vdash}\,\exists_I \qquad \frac{\Gamma \vdash \exists y.A\{x := y\} \quad \Delta, [u : A] \vdash C}{\Gamma, \Delta \vdash C}\,\exists_E$$

- Condition on $\forall_I$:
  - $y \equiv x$ or $y \notin FV(A)$; and
  - y not free in Γ.
- Condition on $\exists_E$:
  - $y \equiv x$ or $y \notin FV(A)$; and
  - y not free in Δ.

3 Sequent calculus

3.1 G1c

$$\frac{}{\underline{A} \vdash \underline{A}}\ Ax \qquad \frac{}{\underline{\bot} \vdash}\ L\bot$$

$$\frac{\Gamma \vdash \Delta}{\underline{A}, \Gamma \vdash \Delta}\ LW \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \underline{A}}\ RW$$

$$\frac{A, A, \Gamma \vdash \Delta}{\underline{A}, \Gamma \vdash \Delta}\ LC \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, \underline{A}}\ RC$$

$$\frac{A_I, \Gamma \vdash \Delta}{\underline{A_0 \wedge A_1}, \Gamma \vdash \Delta}\ L\wedge \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, \underline{A \wedge B}}\ R\wedge$$

$$\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{\underline{A \vee B}, \Gamma \vdash \Delta}\ L\vee \qquad \frac{\Gamma \vdash \Delta, A_I}{\Gamma \vdash \Delta, \underline{A_0 \vee A_1}}\ R\vee$$

$$\frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{\underline{A \supset B}, \Gamma \vdash \Delta}\ L\supset \qquad \frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, \underline{A \supset B}}\ R\supset$$

$$\frac{A\{x := t\}, \Gamma \vdash \Delta}{\underline{\forall x.A}, \Gamma \vdash \Delta}\ L\forall \qquad \frac{\Gamma \vdash \Delta, A\{x := y\}}{\Gamma \vdash \Delta, \underline{\forall x.A}}\ R\forall$$

$$\frac{A\{x := y\}, \Gamma \vdash \Delta}{\underline{\exists x.A}, \Gamma \vdash \Delta}\ L\exists \qquad \frac{\Gamma \vdash \Delta, A\{x := t\}}{\Gamma \vdash \Delta, \underline{\exists x.A}}\ R\forall$$

- Γ, Δ are *side formulae*
- the underlined in the conclusion is the *principal* formula
- y not free in the conclusion in $R\forall$ and $L\exists$

3.2 G2m

$$\frac{}{\Gamma, A \vdash A}\ Ax$$

$$\frac{A, A, \Gamma \vdash B}{A, \Gamma \vdash B}\ LC \qquad \frac{\Gamma \vdash B}{A, \Gamma \vdash B}\ LW$$

$$\frac{A_I, \Gamma \vdash C}{A_0 \wedge A_1, \Gamma \vdash C}\ L\wedge \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ R\wedge$$

$$\frac{A, \Gamma \vdash C \quad B, \Gamma \vdash C}{A \vee B, \Gamma \vdash C}\ L\vee \qquad \frac{\Gamma \vdash A_I}{\Gamma \vdash A_0 \vee A_1}\ R\vee$$

$$\frac{\Gamma \vdash A \quad B, \Gamma \vdash C}{A \supset B, \Gamma \vdash C}\ L\supset \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\ R\supset$$

$$\frac{A\{x := t\}, \Gamma \vdash B}{\forall x.A, \Gamma \vdash B}\ L\forall \qquad \frac{\Gamma \vdash A\{x := y\}}{\Gamma \vdash \forall x.A}\ R\forall$$

$$\frac{A\{x := y\}, \Gamma \vdash B}{\exists x.A, \Gamma \vdash B}\ L\exists \qquad \frac{\Gamma \vdash A\{x := t\}}{\Gamma \vdash \exists x.A}\ R\forall$$

- y not free in the conclusion in $R\forall$ and $L\exists$

3.3 G3i

$$\frac{}{P, \Gamma \vdash P}\,Ax \qquad \frac{}{\bot, \Gamma \vdash C}\,L\bot$$

$$\frac{A, B, \Gamma \vdash C}{A \wedge B, \Gamma \vdash C}\,L\wedge \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\,R\wedge$$

$$\frac{A, \Gamma \vdash C \quad B, \Gamma \vdash C}{A \vee B, \Gamma \vdash C}\,L\vee \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\,R\vee_0 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\,R\vee_1$$

$$\frac{A \supset B, \Gamma \vdash A \quad B, \Gamma \vdash C}{A \supset B, \Gamma \vdash C}\,L\supset \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\,R\supset$$

$$\frac{A\{x := t\}, \forall x.A, \Gamma \vdash C}{\forall x.A, \Gamma \vdash C}\,L\forall \qquad \frac{\Gamma \vdash A\{x := y\}}{\Gamma \vdash \forall x.A}\,R\forall$$

$$\frac{A\{x := y\}, \Gamma \vdash C}{\exists x.A, \Gamma \vdash C}\,L\exists \qquad \frac{\Gamma \vdash A\{x := t\}}{\Gamma \vdash \exists x.A}\,R\exists$$

G3m se obtiene de **G3i** eliminando el axioma $L\bot$.

3.4 G3c

$$\frac{}{P, \Gamma \vdash \Delta, P}\,Ax \qquad \frac{}{\bot, \Gamma \vdash \Delta}\,L\bot$$

$$\frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\,L\wedge \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\,R\wedge$$

$$\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}\,L\vee \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}\,R\vee$$

$$\frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta}\,L\supset \qquad \frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B}\,R\supset$$

$$\frac{A\{x := t\}, \forall x.A, \Gamma \vdash \Delta}{\forall x.A, \Gamma \vdash \Delta}\,L\forall \qquad \frac{\Gamma \vdash \Delta, A\{x := y\}}{\Gamma \vdash \Delta, \forall x.A}\,R\forall$$

$$\frac{A\{x := y\}, \Gamma \vdash \Delta}{\exists x.A, \Gamma \vdash \Delta}\,L\exists \qquad \frac{\Gamma \vdash \Delta, \exists x.A, A\{x := t\}}{\Gamma \vdash \Delta, \exists x.A}\,R\exists$$